B. Math. III and M. Math. I Topology Final Examination 2011

Each question carries 12 marks. Attempt all questions. Anything proved in the class maybe cited without proof. Results of exercises, however, must be derived in full. You may use books and notes.

1. (i): Let S_Ω denote the first uncountable ordinal space, and let $A \subset S_\Omega$ be a non-empty subset. Let a be the smallest element of A. Show that the singleton set $\{a\}$ is both open and closed in A (Here A is equipped with subspace topology).

   (ii): Let X be a connected topological space, and $f : X \to S_\Omega$ be a continuous map. Show that f is a constant map.

2. (i): Let $p : X \to Y$ be a continuous map of topological spaces. Suppose there exists a continuous map $f : Y \to X$, such that $p \circ f = \mathrm{id}_Y$. Show that p is a quotient map.

   (ii): Let (X, d) be a compact metric space, and $f : X \to X$ a continuous map such that :
   $$d(f(x), f(y)) < d(x, y) \quad \text{for all } x \neq y \text{ in } X$$
   Show that there exists a fixed point x for f (i.e. x satisfies $f(x) = x$).

3. (i): Let $\mathbb{R}_l$ denote the set of reals with the lower limit topology. Show that $\mathbb{R}_l$ is separable.

   (ii): Show that $\mathbb{R}_l$ is not metrizable.

4. (i): Let X be a connected normal topological space with card $X \geq 2$. Show that X is uncountable.

   (ii): Let $M(n, \mathbb{C})$ denote the space of all $n \times n$ matrices, with its usual metric topology (regarding it as the Euclidean space $\mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$). Prove that the subspace :
   $$GL(n, \mathbb{C}) = \{A \in M(n, \mathbb{C}) : \det A \neq 0\}$$
   is path connected. (*Hint:* Let I denote the identity matrix. The map $p : \mathbb{C} \to \mathbb{C}$ defined by the polynomial $p(z) = \det(I + z(A - I))$ has only finitely many zeros.)

5. (i): Prove that there is no continuous function $h : \mathbb{C} \to \mathbb{C}$ satisfying $(h(z))^3 = z$ for all $z \in \mathbb{C}$. (Contrast with $\mathbb{R}$, where the cube root function $h(x) = x^{1/3}$ is a continuous function satisfying $(h(x))^3 = x$ for all $x \in \mathbb{R}$).

   (ii): Show that a continuous map $f : \mathbb{RP}^2 \to S^1 \times S^1$ is nullhomotopic (i.e. homotopic to a constant map).